

P_3 -GAMES

Wing-Kai Hon¹, Ton Kloks, Fu-Hong Liu¹, Hsiang-Hsuan Liu^{1,2}, and
Tao-Ming Wang³

¹ National Tsing Hua University, Hsinchu, Taiwan
(wkhon, fhliu, hhliu)@cs.nthu.edu.tw

² University of Liverpool, Liverpool, United Kingdom
hhliu@liverpool.ac.uk

³ Tunghai University, Taichung, Taiwan
wang@go.thu.edu.tw

Abstract. Without further ado, we present the P_3 -game. The P_3 -game is decidable for elementary classes of graphs such as paths and cycles. From an algorithmic point of view, the connected P_3 -game is fascinating. We show that the connected P_3 -game is polynomially decidable for classes such as trees, chordal graphs, ladders, cacti, outerplanar graphs and circular arc graphs.

1 Introduction

Definition 1. An alignment is a pair (X, \mathcal{L}) , where X is a finite set and \mathcal{L} is a collection of subsets of X satisfying

1. $\emptyset \in \mathcal{L}$ and $X \in \mathcal{L}$, and
2. \mathcal{L} is closed under intersections.

The elements of the set \mathcal{L} of an alignment are called ‘closed.’

Definition 2. Let (X, \mathcal{L}) be an alignment and let $A \subseteq X$. The hull of A , denoted $\sigma(A)$, is defined as

$$\sigma(A) = \bigcap_{K \in \mathcal{L} \text{ and } A \subseteq K} K.$$

The hull operator σ is a closure operator, that is, it satisfies

- (i) $\sigma(\emptyset) = \emptyset$,
- (ii) $S \subseteq \sigma(S)$ for each set S ,
- (iii) For sets S and T , if $S \subseteq T$ then $\sigma(S) \subseteq \sigma(T)$, and
- (iv) for each set S , $\sigma(\sigma(S)) = \sigma(S)$.

Definition 3. Let G be a graph. A set $S \subseteq V(G)$ is P_3 -closed if

$$\forall_{x \in V(G)} \quad x \notin S \quad \Rightarrow \quad |N(x) \cap S| < 2.$$

For a graph G , the pair $(V(G), \mathcal{L})$, where \mathcal{L} is the collection of P_3 -closed sets in G , is an alignment.

Two players play the P_3 -game on a graph by alternately selecting vertices. At the start of the game all vertices are unlabeled. During the game players label vertices. Prior to every move, the set of labeled vertices is P_3 -closed. Let L denote the set of labeled vertices. A move consists of labeling a, previously unlabeled, vertex x . In effect, the new set of labeled vertices becomes $\sigma(L + x)$.

According to the Sprague-Grundy theory, when the P_3 -game is played on a finite graph, there is a winning strategy for one of the two players. If there is a polynomial-time algorithm to decide whether there is a winning strategy for one of the two players, we call the game decidable (in polynomial time). For example, when the graph is a clique with at least two vertices, then the second player wins the game. That is so because the convex hull of any two vertices in a clique is $V(G)$.

Another example where the game is easy to decide is the case where the playground is a star.

Lemma 1. *Assume the graph is a star, $K_{1,t}$. Then player one has a winning strategy if and only if the number of leaves, t , is even.*

Proof. Assume the graph is a star with an even number of leaves. A winning move for player one is to choose the center.

Assume the playground is a star with an odd number of leaves. The winning strategy for player two is to choose a vertex which leaves the playground with an even number of unlabeled leaves. \square

In his acclaimed paper on monophonic alignments, Duchet defines a *graph-alignment* as a pair (G, \mathcal{C}) , where G is a connected graph and \mathcal{C} is a collection of subsets of $V(G)$ such that $(V(G), \mathcal{C})$ is an alignment and the following additional property holds.

Every member of \mathcal{C} induces a connected subgraph of G .

The legal moves in the connected P_3 -game are restricted such that the P_3 -closed sets induce connected subgraphs. Prior to every move, the playground is a connected graph G , with a set L of labeled vertices satisfying

- (1) L is P_3 -closed, and
- (2) $G[L]$, that is the subgraph induced by L , is connected.

A legal move in the connected P_3 -game is the selection of an unlabeled vertex x such that $\sigma(L + x)$ induces a connected graph. In other words, a move is legal if the selected vertex is at distance at most two from the set L .

2 The connected P_3 -game on trees

To analyze a game played on graphs, one makes use of the ‘game graph.’ This is a directed graph (P, Γ) , constructed as follows. Each node in the game graph represents a playground, and there is an arc $(u, v) \in \Gamma$ if the playground corresponding with the node v can be reached from the playground corresponding with u in one move.

The Grundy function $g : P \rightarrow \mathbb{N} \cup \{0\}$ is defined on P as follows. If $p \in P$ is a sink, that is, a node without outgoing arcs, the Grundy value is defined as $g(p) = 0$. For any other node, say q , the Grundy value is defined as

$$g(q) = \min \{ n \mid n \in \mathbb{N} \cup \{0\} \text{ and } \forall_p (q, p) \in \Gamma \Rightarrow g(p) \neq n \}.$$

Thus, $g(q)$ is the smallest nonnegative integer which is not attained by any node in P that can be reached from q in one move. We allude that the Grundy function is sometimes called the mex-function, which stands for *minimal excluded value*.

Let $s \in P$ be the initial playground, before any move has been made. The Grundy value $g(s)$ is the Grundy value of the game P .

The following theorem is easy to check.

Theorem 1. *Let $s \in P$ be the node representing the initial playground. Then player one has a winning strategy if and only if $g(s) \neq 0$.*

Proof. Notice that, from any node q with $g(q) \neq 0$, a player can move to a node $p \in P$ with $g(p) = 0$. □

Consider a finite collection of games G_1, \dots, G_t . The product game

$$G = G_1 \times \dots \times G_t$$

is the game where a player is allowed to make a move in *one* of the games G_i .

Let g_i denote the Grundy value for game G_i . The nim-sum $g_1 \oplus \dots \oplus g_t$ is obtained as follows. Write each Grundy value g_i in binary and add them up, *without a carry*. Thus, for example, $3 \oplus 6 = 5$. Grundy’s theorem is the following.

Theorem 2 (Grundy’s Theorem). *The Grundy value g for the product game G is*

$$g = g_1 \oplus \dots \oplus g_t.$$

Theorem 3. *There exists a polynomial-time algorithm that decides if the first player has a winning strategy in the connected P_3 -game played on a tree.*

Proof. Let T be a tree. Consider a playground p , which is characterized by a connected subtree of T . A branch is a subtree with a maximal number of edges, which contains one node of the subtree p as a leaf. We call the leaf the root of the branch. Let B_1, \dots, B_ℓ be the branches that have at least one edge. Denote

the Grundy value of B_i as g_i . The Grundy value of p is the Grundy value of $B_1 \times \cdots \times B_\ell$. By Grundy's Theorem 2, this is the nim-sum of the Grundy values g_i , that is,

$$g(p) = g_1 \oplus \cdots \oplus g_\ell. \quad (1)$$

For each possible initial move s of the first player, the algorithm computes the Grundy value $g(s)$, via Equation (1), by dynamic programming on the branches.

By Theorem 1, this proves the theorem. \square

3 The P_3 -game on paths and cycles

Paths

Theorem 4. *There exists an $O(n^2)$ algorithm that decides the P_3 -game on paths.*

Proof. Consider the set of paths with 1 up to n vertices. Create a set in which each path occurs 4 times as it is bordered by a labeled vertex on either side or not. So, the set contains $4n$ elements. For each element, the Grundy value is computed by considering all possible moves. Each move divides the path into at most two, strictly smaller elements. The algorithm processes the paths in the set in order of increasing length. \square

Remark 1. As for now, we don't have an easy formula which tells whether P_n is won for player one.

Cycles

Theorem 5. *There exists a polynomial-time algorithm to decide the P_3 -game on cycles.*

Proof. When the cycle has an even number of vertices, the winning strategy for player two is to choose the vertex opposite player one's choice.

The odd case is, unless it is K_3 , less trivial. We can reduce it to paths and use Theorem 4 as follows. Consider a cycle with an odd number $n > 3$ vertices. The first player selects some arbitrary vertex. Split the selected vertex in two vertices, creating a path with two selected vertices at the ends, and $n - 1$ unselected vertices between them. According to Theorem 4, there is an $O(n^2)$ algorithm to decide this game. \square

4 The connected P_3 -game on paths and cycles

Cycles

Theorem 6. *The first player has a winning strategy when the game is played on a cycle C_n if and only if $n \equiv 2 \pmod{3}$.*

Proof. We show how to calculate the Grundy value. Consider a cycle with n vertices. Let $f(k)$ be the Grundy value when the playground is a path in C_n with k vertices. Then $f(n) = 0$.

There cannot be a playground with one vertex not selected, thus we need to leave $f(n-1)$ undefined. Next, we have $f(n-2) = 1$ since the game ends in the next move when two vertices are not selected. Subsequently, we find

$$\begin{aligned} f(n-3) &= \text{mex} \{ f(n-2), f(n) \} = 2, \\ f(n-4) &= \text{mex} \{ f(n-3), f(n-2) \} = 0, \\ f(n-5) &= \text{mex} \{ f(n-4), f(n-3) \} = 1, \quad \&tc. \end{aligned}$$

It is now easy to prove, by induction, that, for $i \geq 2$,

$$f(n-i) = \begin{cases} 0 & \text{if } i \equiv 1 \pmod{3} \\ 1 & \text{if } i \equiv 2 \pmod{3} \\ 2 & \text{if } i \equiv 0 \pmod{3}. \end{cases}$$

This implies that

$$f(1) = \begin{cases} 0 & \text{if } n \equiv 2 \pmod{3} \\ 1 & \text{if } n \equiv 0 \pmod{3} \\ 2 & \text{if } n \equiv 1 \pmod{3}. \end{cases}$$

Let $g(n)$ be the Grundy value for the cycle with n vertices. Then

$$g(n) = \text{mex} \{ f(1) \} = \begin{cases} 1 & \text{if } n \equiv 2 \pmod{3} \\ 0 & \text{otherwise.} \end{cases}$$

This proves the theorem. □

Paths We denote a path with n vertices as P_n .

Theorem 7. *The first player has a winning strategy in the connected P_3 -game played on a path with n vertices if $n \neq 2$.*

Proof. Let $f(n)$ denote the Grundy value for the path with one of the endpoints labeled. We claim that

$$f(n) = \begin{cases} 0 & \text{if } n \equiv 1 \pmod{3} \\ 1 & \text{if } n \equiv 2 \pmod{3} \\ 2 & \text{if } n \equiv 0 \pmod{3}. \end{cases}$$

This claim is readily checked via the recurrence

$$f(n) = \text{mex} \{ f(n-1), f(n-2) \}.$$

Let $g(n)$ be the Grundy value for the path $p(n)$. Then we have that

$$g(n) = \text{mex} \{ f(n), f(n-i) \oplus f(i+1) \mid 1 \leq i < n-1 \}.$$

It follows that, for $n \neq 2$,

$$g(n) = \begin{cases} \text{mex} \{ 0, 3 \} = 1 & \text{if } n \equiv 1 \pmod{3} \\ \text{mex} \{ 0, 1 \} = 2 & \text{if } n \equiv 2 \pmod{3} \\ \text{mex} \{ 0, 2 \} = 1 & \text{if } n \equiv 0 \pmod{3}. \end{cases}$$

This proves the theorem. \square

5 The P_3 -game on cographs

A P_4 denotes a path with four vertices.

Definition 4. A *cograph* is a graph without induced P_4 .

One characterization of cographs is that, every induced subgraph with at least two vertices is either a join or a union of two smaller cographs.

Theorem 8. *There exists an algorithm to decide the P_3 -game in polynomial time on cographs.*

Proof. Let G be a cograph. When G is disconnected, the game reduces to the sum of the games played on the components of G . By the Sprague-Grundy theorem, the Grundy value of the game is the nim-sum of the Grundy values played on the components.

Assume that G is connected. Then G is the join of two smaller cographs G_1 and G_2 , that is, all vertices of G_1 are adjacent to all vertices of G_2 . The algorithm considers all possible playgrounds after both players have made a move. First assume that both players chose a vertex of G_1 . Then all vertices of G_2 are added to the hull after the second move. When G_2 has at least two vertices, then the game is over, since all vertices of G_1 are also in the P_3 -closure.

Assume that G_2 has only one vertex. Notice that this vertex is universal, that is, it is adjacent to all other vertices. Let C_1, \dots, C_t be the components of G_1 . The components that contain x and y are subsets of the hull. The components that do not contain x nor y are added one by one in every subsequent move. It follows that, player one wins the game from this position if and only if the number of components that do not contain x or y is odd.

Assume that player one makes his first move in G_1 and that player two makes his first move in G_2 . Then the next move adds a vertex of G_1 or G_2 and this reduces the analysis to one of the previous cases.

This proves the theorem. \square

6 The connected P_3 -game on ladders

Definition 5. A ladder is the Cartesian product of two paths, one of which has only one edge.

A ladder is denoted as $L_n = P_2 \times P_n$ and it has $2n$ vertices and $2(n - 1) + n$ edges. It consists of two paths P_n of length $n - 1$ and a perfect matching. The edges of the matching are called the rungs and the two paths P_n are called the ringers, or rails or stiles.

Theorem 9. The first player has a winning strategy for the connected P_3 -game on a ladder L_n if and only if $n \equiv 0 \pmod{6}$.

Proof. First assume that $n \equiv 0 \pmod{6}$. Partition the ladder into dominoes, that is, $P_2 \times P_3$. To win the game, player one starts in a corner of the ladder, that is, a vertex of degree two. It is easy to check that, no matter what player two plays, player one always completes the first, and then every subsequent domino of the ladder.

Now assume that $n \not\equiv 0 \pmod{6}$. Assume player one starts in a corner. Then player two chooses a vertex in the other stile, such that the remaining rungs can be partitioned into dominoes. Then, when player one enters a domino, player two can choose a vertex which encloses the domino in the P_3 -closure.

Assume player one starts in some middle rung. Player two chooses a vertex in the other stile, such that the two numbers of rungs below and above the P_3 -closure are equivalent modulo 3. Each next move adds either one or two rungs to the P_3 -closure. The strategy of player two is to keep the numbers of remaining rungs modulo three, below and above the P_3 -closure the same. \square

7 The connected P_3 -game on caterpillars

A caterpillar is a tree that contains a dominating path, that is, a path such that every other vertex is connected to a vertex in the path. Alternatively, a caterpillar is a tree without the subdivision of a star $K_{1,3}$ as a subgraph. The algorithm that decides the connected P_3 -game for trees simplifies a little bit in case the tree is a caterpillar. In this section we shortly describe this simplification.

The dominating path of the caterpillar is called the backbone and we denote it by P . Assume that P is a path with n vertices, $P \simeq P_n$. We assume that the endpoints of the backbone are leaves of the caterpillar. The vertices of the backbone are numbered, $1, \dots, n$. Let the number of feet adjacent to the point i be denoted as $h(i)$.

Theorem 10. There exists an efficient algorithm to decide if the first player has a winning strategy in the connected P_3 -game played on a caterpillar.

Proof. We analyze all possible positions after two moves have been made. First assume that the first and second move are the selection of two feet x and y adjacent to the same vertex i in the P . In that case, the P_3 -closure is the $P_3, \{x, y, i\}$. The remaining game is split into a caterpillar with backbone $P[[1, \dots, i]]$, one caterpillar with backbone $P[[i, \dots, n]]$, and $h(i) - 2$ edges with i as an endpoint. The Grundy value for this position is the nim-sum of the games described above, with initially labeled vertex i . Note that the nim-sum of the $h(i) - 2$ edges with labeled vertex i is

$$\bigoplus_{\ell=1}^{h(i)-2} 1 = \begin{cases} 1 & \text{if } h(i) \text{ is even} \\ 0 & \text{otherwise.} \end{cases}$$

Assume that the first move is the selection of a foot x adjacent to i , and the second move is the selection of the vertex i . The remaining game is split into a game on a caterpillar with backbone $P[[1, \dots, i]]$, one caterpillar with backbone $P[[i, \dots, n]]$, and $h(i) - 1$ games on edges with endpoint i .

Assume that the first move is the selection of a foot x adjacent to i and the second move is the selection of the vertex $i + 1$. The P_3 -closure is $\{x, i, i + 1\}$, and the game is split into a game on the caterpillars with backbones $P[[1, \dots, i]]$ and $P[[i + 1, \dots, n]]$, the $h(i) - 1$ edges consisting of the remaining leaves adjacent to i and the $h(i + 1)$ edges connecting leaves to $i + 1$.

The first two moves are the selection of vertices i and $i + 1$. The game is split into two caterpillars, one with backbone $P[[1, \dots, i]]$ and one with backbone $P[[i + 1, \dots, n]]$, and $h(i)$ and $h(i + 1)$ edges connecting leaves with i and $i + 1$ respectively.

The remaining case is where two vertices i and $i + 2$ of the backbone are selected. The game is split, similar as to that described above.

It follows that the game can be decided by an algorithm that performs dynamic programming on caterpillars with backbones that are subpaths of P with one end a labeled vertex of P and the other end the point 1 or n . This algorithm can be implemented to run in $O(n)$ time; precisely speaking, it is the number of these subpaths of P .

This proves the theorem. □

8 The connected P_3 -game on chordal graphs

Definition 6. A graph is chordal if it has no induced cycle of length more than three.

The following lemma is easily checked. For a proof see, eg, Centeno et al.

Lemma 2. If x and y are two vertices at distance at most two in a biconnected chordal graph G , then

$$\sigma(\{x, y\}) = V(G).$$

Theorem 11. *There exists a polynomial-time algorithm to decide the connected P_3 -game on connected chordal graphs.*

Proof. The algorithm considers all possible initial playgrounds after the first two moves have been made. If the graph is not connected, then the vertices that are not in the initial playground, are partitioned into components for which the neighborhood is a cutvertex. Consider the components, rooted at the cutvertices. By the Sprague-Grundy theorem, the Grundy value of the playground is the nim-sum of the games played at these components. It follows that this is computable in polynomial time by dynamic programming on the biconnected components.⁴ \square

9 The connected P_3 -game on cacti

In this section we prove that the connected P_3 -game on cacti is decidable in polynomial time.

Definition 7. *A cactus is a graph without the diamond as a minor.*

Equivalently, a graph is a cactus if every edge is a subset of at most one cycle of the graph. Also, a graph is a cactus if every biconnected component is an edge or a cycle.

Theorem 12. *There exists a polynomial-time algorithm to decide the connected P_3 -game on cacti.*

Proof. We show that the Grundy value is computable in polynomial time.

The algorithm tries all possible vertices as a first move for player one. Consider a vertex x as a first move. Assume that x is in a cycle and let R be a cycle containing x . First assume that x is a cutvertex. Consider the components of $G - x$ augmented with the vertex x . By the Sprague-Grundy theorem, the Grundy value is the nim-sum of the Grundy values of the games played on the augmented components.

For the augmented component that contains R , the children of the playground $\{x\}$ are the playgrounds that are edges that have x as an endpoint and those that are P_3 's with x as an endpoint. For each of these playgrounds, if it contains a cutvertex y , the game is split into components of $G - y$ augmented with y . The Grundy value is the nim-sum of these games. For each of the components that do not contain vertices of $R \setminus \{y\}$, the algorithm recursively computes the Grundy value. Notice that the initial playground for each component is either a single vertex y , or an edge incident with y . When it is an edge, the playground extends greedily into maximal biconnected chordal subgraph.

⁴ For the definition and properties of biconnected component, we refer the readers to https://en.wikipedia.org/wiki/Biconnected_component.

There are only $O(|R|^2)$ different playgrounds induced on R . For each playground r on R the algorithm computes the mex value from its children. If a child includes a cutvertex, the game is split. For the augmented components that share a cutvertex with R , with a playground that is either the cutvertex, or an edge incident with the cutvertex, the Grundy value is computed recursively. \square

10 The connected P_3 -game on outerplanar graphs

A graph is outerplanar if it has a plane embedding with all vertices lying on the outerface. Alternatively, outerplanar graphs are defined as follows.

Definition 8. *A graph is outerplanar if it does not contain K_4 or $K_{2,3}$ as a minor.*

Theorem 13. *There exists a polynomial-time algorithm that decides the connected P_3 -game on outerplanar graphs.*

Proof. The proof is similar to that for the cacti. For ease of description, we assume that G is a biconnected outerplanar graph. The algorithm that we describe extends in an obvious manner for cases when G contains cutvertices. When G is biconnected, it forms a ‘tree of cycles,’ which can be defined recursively as follows. Any cycle is a biconnected outerplanar graph. A biconnected outerplanar graph G' , with an outerface O , and a cycle C , a new biconnected outerplanar graph is formed by identifying the endpoints of an edge with the endpoints of an edge in O . This recursively defines all biconnected outerplanar graphs.

All minimal separators are edges and for each edge e , $G - e$ contains exactly two components. We call the components with the edge e added to it, the augmented components at the edge e .

For each separator e , and for each augmented component C at e , the algorithm recursively computes the Grundy value for the playground that consists of e , and for the playground consisting of e plus one additional vertex adjacent to an endpoint of e . When the cycle incident with e is a triangle, the playground greedily extends.

To process G , the algorithm tries all vertices x as an initial playground p . The children of p are edges incident with x and P_3 's with x as an endpoint. When a child contains a minimal separator, the game splits into two games played on the augmented components. In that case, the Grundy value is the nim-sum of the two games on the augmented components.

Consider a playground p that contains only edges of the outerface O . Then p is a simple path. The children are all extensions of p with an edge or a P_3 . An extension q that contains a minimal separator splits the game, and the Grundy value is the nim-sum of the two augmented components, each with the induced paths q_1 and q_2 as a playground.

For the children q that extend p with edges along O , the Grundy value is computed by dynamic programming. Finally, for the playground p , the Grundy value is computed as the mex function of the Grundy values of its children.

This proves the theorem. \square

11 The connected P_3 -game on circular arc graphs

Gavril initiated the research on circular-arc graphs. These graphs are the intersection graphs of arcs on a circle. McConnell showed that this class can be recognized in linear time. Whilst intervals on the real line satisfy the Helly property, this is no longer true for circular arcs on a circle. That is, there could be a triangle in the graph without any point on the circle that is in all three arcs.

Gavril defines a Helly circular-arc graph as a graph for which the clique matrix has the circular 1s property for columns. Also the Helly circular-arc graphs are recognizable in linear time.

Definition 9. *A graph is a Helly circular-arc graph if it has a circular maximal clique arrangement.*

Theorem 14. *There exists a polynomial-time algorithm to decide the connected P_3 -game on Helly circular-arc graphs.*

Proof. Obviously, we may assume that the graph is connected.

To reduce the circular problem to a linear one, the algorithm considers all possible first two moves.

When the first two vertices are selected, these two vertices and their common neighbors are labeled. The closure recursively labels all vertices in cliques that contain two previously labeled vertices.

There is at least one maximal clique in which all vertices are labeled, and so, the graph can be regarded as an interval graph, i.e. a graph that has a linear order for its maximal cliques.

There may exist one or two vertices that appear in (consecutive) maximal cliques on the lefthand side of the linear order and in the maximal cliques on the right-hand side. These vertices do not appear in consecutive maximal cliques, but, since they are labeled, they can be ignored in subsequent procedures.

The vertices that are not in the initial playground are partitioned into two types of components:

- (1) the components that have a neighborhood consisting of a single labeled cutvertex, and
- (2) the components of which the neighborhood consists of two labeled cutvertices that border the consecutive cliques of the component.

For a component of the first type, it takes exactly one additional move to label all its vertices. Components of the second type are handled recursively by considering all possible extra moves, rooted at distance at most two from either the cutvertex on the righthand side or the lefthand side.

This shows that the Grundy values of components can be computed recursively. By dynamic programming, the program calculates the Grundy value of any sequence of components. The Grundy value for the circular arc graph is computed by considering all possible first two moves. \square

With a different analysis, the algorithm for Helly circular-arc graphs would accommodate general circular-arc graphs. Instead of considering max cliques when analyzing the first two moves, we consider *Helly cliques*, a maximal set of arcs overlapping in one point of the circle. After the selection of the first two vertices, there is at least one Helly clique in which all vertices (arcs) are labeled. For any such labeled Helly clique, it cuts the circle with respect to the labeling so that the circle becomes an arc A . The rest of the input arcs (on A) can be regarded as intervals on a line. Thus the graph of the rest of input becomes an interval graph and the algorithm for Helly circular-arc graphs proceeds.

Theorem 15. *There exists a polynomial-time algorithm to decide the connected P_3 -game on general circular-arc graphs.*

12 Concluding remark

We are aware of only very few problems that are solvable in polynomial time on outerplanar graphs, and that resist efficient algorithms for graphs of treewidth two. At the moment we do not have an efficient algorithm to decide the connected P_3 -game for graphs of treewidth two.

We are interested in the connected P_3 -game on convex geometries. These are alignments satisfying the anti-exchange property. We are primarily interested in the question under which additional conditions the connected P_3 -game becomes decidable on convex geometries (see [2,4,6]).

References

1. Centeno, C., M. Dourado, L. Penso, D. Rautenbach and J. Szwarcfiter, Irreversible conversion of graphs, *Theoretical Computer Science* **412** (2011), pp. 3693–3700.
2. Chvátal, V., Antimatroids, betweenness, convexity. In (W. Cook, L. Lovász and J. Vygen eds.) *Research trends in combinatorial optimization*, Bonn 2008, Springer-Verlag, Berlin, pp. 57–74.
3. Duchet, P., Convex sets in graphs, II. Minimal path convexity, *Journal of Combinatorial Theory, Series B*, **44** (1988), pp. 307–316.
4. Edelman, P. and R. Jamison, The theory of convex geometries, *Geometriae Dedicata* **19** (1985), pp. 247–270.

5. Gavril, F., Algorithms on circular-arc graphs, *Networks* **4** (1974), pp. 357–369.
6. Grier, D., Deciding the winner of an arbitrary finite poset game is PSPACE-complete, *Proceedings ICALP'13*, Springer-Verlag, LNCS 7965 (2013), pp. 497–503.
7. Grundy, P., Mathematics and games, *Eureka* **2** (1939), pp. 6–8.
8. Joeris, B., M. Lin, R. McConnell, J. Spinrad and J. Szwarcfiter, Linear-time recognition of Helly circular-arc models and graphs, *Algorithmica* **59** (2009), pp. 215–239.
9. Kloks, T. and Y. Wang, *Advances in graph algorithms*. Manuscript on ViXrA:1409.0165, 2014.
10. Korte, B., L. Lovász and R. Schrader, *Greedoids*, Springer-Verlag, Berlin, 1990.
11. McConnell, R., Linear-time recognition of circular-arc graphs, *Algorithmica* **37** (2013), pp. 93–147.
12. Nienhuys, J., Graphs and Games. In (L. Hung and T. Kloks eds.) *De Bruijn's Combinatorics — Classroom notes*. Manuscript on ViXrA:1208.0223, 2012.